Hamiltonian Truncation in 1D Long-Range CFTs

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Outline

Introduction

Motivation

- Overarching goal: solve QFT in the strong coupling limit

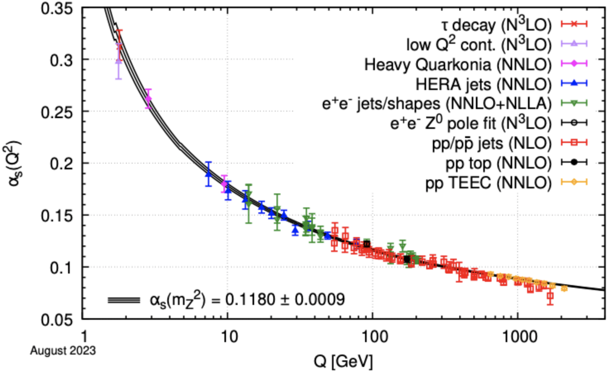
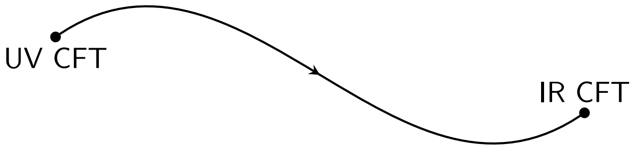


Figure: QCD running coupling (Navas et al. 2024)

Motivation

- First, tackle CFT in the strong coupling limit



- Two and three-point functions fully constrained $\rightarrow$ CFT data $\{\Delta_i, \lambda_{jkl}\}$ + OPE $\rightarrow$ conformal bootstrap

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}} C_{12\mathcal{O}}(x,\partial)\mathcal{O}(0) \tag{1.1}$$

Why long-range CFTs

- Conformal symmetry in 1D $\rightarrow$ stress tensor vanishes in local theories! Given a local operator $\mathcal{O}(x)$:

$$[T, \mathcal{O}(x)] = 0 \text{ since } T = 0 \tag{1.2}$$

- Statement lifts to position-independence of observables
- Non-trivial correlators in 1D CFT require absence of a stress tensor $\rightarrow$ non-local theory

Long-range Ising and Lee-Yang models

Long-Range Ising Model

$$S_{\mathrm{LORI}}[\phi] = \frac{\mathcal{K}_\sigma}{2} \int \mathrm{d}^d x \mathrm{d}^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} + \frac{\lambda}{4!} \int \mathrm{d}^d x \phi^4(x) \qquad (2.1)$$

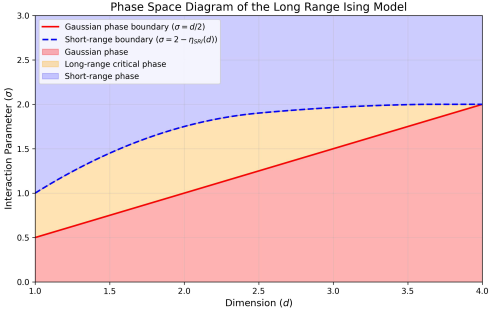


Figure: Phase diagram of LORI (figure from Bianchi, LSC, and de Sabbata 2024). Proof of conformal invariance in Paulos et al. 2016.

Long-Range Lee-Yang Model

$$S_{\text{LORALY}}[\phi] = \frac{\mathcal{K}_\sigma}{2} \int \mathrm{d}^d x \mathrm{d}^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} + \frac{ig}{3!} \int \mathrm{d}^d x \phi^3(x) \quad (2.2)$$

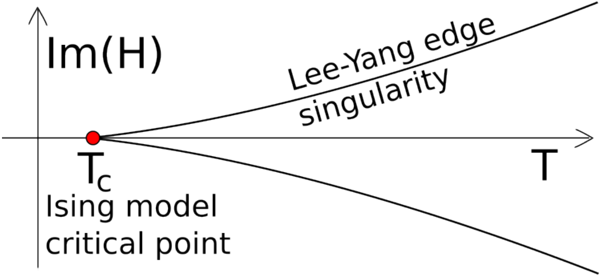


Figure: (Wikipedia) Zeros of the partition function of the Ising model (Yang and Lee 1952; Lee and Yang 1952; Cardy 2023)).

AdS perspective

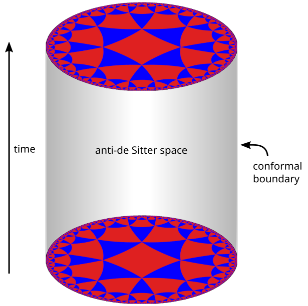


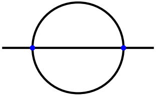
Figure: (Wikipedia) ∂AdS is conformal to a cylinder.

- Massive local scalar field in the bulk
- Integrate out bulk $\rightarrow$ Non-local action on ∂AdS (Witten 1998):

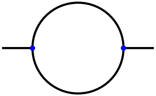
$$S \sim \int \mathrm{d}^d x \mathrm{d}^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} + \text{boundary interactions} \qquad (2.3)$$

ε-expansion

- Set scaling dimension $\Delta_\phi = \frac{d-\varepsilon}{4}$ (LORI), $\frac{d-\varepsilon}{3}$ (LORALY)
- $\lambda_* \propto \varepsilon$ and $g_* \propto \sqrt{\varepsilon}$
- Non-local GFF action $\rightarrow$ no wavefunction renormalization for generic d!



$$\propto \Gamma\left(-\frac{d}{4}\right) + O(\varepsilon) \quad \text{(LORI)} \tag{2.4}$$



$$\propto \Gamma\left(-\frac{d}{6}\right) + O(\varepsilon) \quad \text{(LORALY)} \tag{2.5}$$

- Can recover local theory for $d \rightarrow 4$ or 6 (Bianchi, LSC, and de Sabbata 2024)

Hamiltonian Truncation

General idea

- Determine the spectrum of

$$H = H_0 + V = H_0 + g : (\mathcal{O}_L(0) + \mathcal{O}_R(0)) : \tag{3.1}$$

- Normal-ordered operators on $\mathbb{R} \times S^0$ cylinder

$$\mathbb{R} \times \{-1\} \quad \mathbb{R} \times \{+1\}$$

- H_0 can be a free theory, a solvable 2D CFT...
- Decompose Hilbert space into low and high-energy subspaces $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_h$ using some cutoff Δ_T
- Approximate low-energy spectrum via diagonalization of $H_{|\mathcal{H}_l}$

Radial Quantization

- Natural choice of H_0 for CFT, the dilatation operator:

$$H_0 = D = \sum_n (\Delta + n) a_n^\dagger a_n \tag{3.2}$$

- Field operators on the cylinder:

$$\begin{aligned}
\phi_R(\tau) &= \sum_{n=0}^{\infty} \sqrt{\frac{(2\Delta_\phi)_n}{n!}} \left[e^{-(\Delta_\phi + n)\tau} a_n + e^{(\Delta_\phi + n)\tau} a_n^\dagger \right] \\
\phi_L(\tau) &= \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{(2\Delta_\phi)_n}{n!}} \left[e^{-(\Delta_\phi + n)\tau} a_n + e^{(\Delta_\phi + n)\tau} a_n^\dagger \right]
\end{aligned} \tag{3.3}$$

- Fock basis:

$$|\vec{k}\rangle = \prod_{i=0}^{\infty} \frac{1}{\sqrt{k_i!}} (a_{n_i}^\dagger)^{k_i} |0\rangle \tag{3.4}$$

Radial Quantization

- Matrix elements:

$$\langle \vec{k'} | : \phi_L^n(0) + \phi_R^n(0) : | \vec{k} \rangle$$

$$= 2n! \sum_{\substack{\sum_r (m_r + s_r) = n \\ \sum_r r(m_r+s_r) \in 2\mathbb{N}}} \delta_{\vec{k'}, \vec{k} - \vec{m} + \vec{s}} \prod_r \frac{1}{m_r! s_r!} \sqrt{\left[\frac{(2\Delta_\phi)_r}{r!}\right]^{s_r + m_r} (k_r - m_r + 1)_{m_r} (k_r - m_r + 1)_{s_r}} \tag{3.5}$$

- Can organize Fock states by parity or $\mathbb{Z}_2$ sectors $\rightarrow$ block diagonalization

Tuning to the fixed point

- Truncate spectrum at Δ_T, work at a given Δ_ϕ
- Include all relevant symmetry-preserving counter-terms

$$H \to H_0 + \sum_{\mathcal{O}} g_{\mathcal{O}} : (\mathcal{O}_L(0) + \mathcal{O}_R(0)) : \tag{3.6}$$

- Read off dimension of ϕ, $\Delta_\phi^{(1)}$, namely first $\mathbb{Z}_2$-odd, parity-even state $\to$ in general $\Delta_\phi^{(1)} \neq \Delta_\phi$
- Write as many conformality constraints as there are relevant couplings

$$\begin{cases} \Delta_{\phi^{n-1}} = 1 - \Delta_\phi^{(1)} & \phi^n\text{-theory e.o.m.} \\ \Delta_{\partial^m \phi} = \Delta_\phi^{(1)} + m & \text{existence of some descendants} \end{cases} \tag{3.7}$$

Case of LORI/LORALY

- Normal-ordered interacting Hamiltonian:

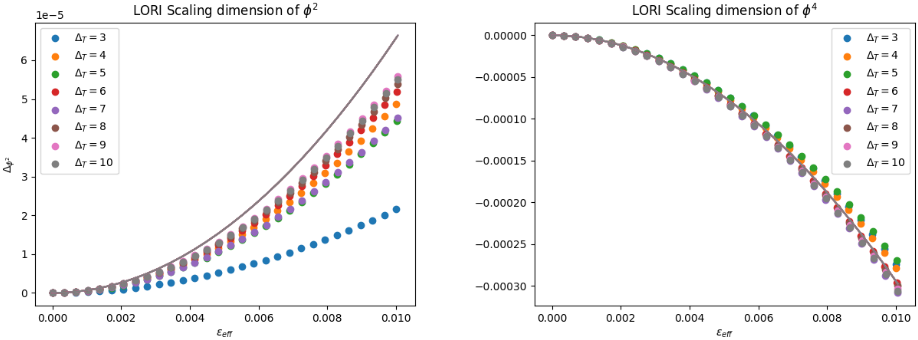
$$H_{\mathrm{LORI}} = H_0 + \frac{\lambda}{4!} : \left(\phi_L^4(0) + \phi_R^4(0)\right) : + g_2 \left(\phi_L^2(0) + \phi_R^2(0)\right) : + \Lambda_4 \mathbb{1} \tag{3.8}$$

$$H_{\mathrm{LORALY}} = H_0 + \frac{ig}{3!} : \left(\phi_L^3(0) + \phi_R^3(0)\right) : + g_2 \left(\phi_L^2(0) + \phi_R^2(0)\right) : + \Lambda_3 \mathbb{1} \tag{3.9}$$

- Constraints:

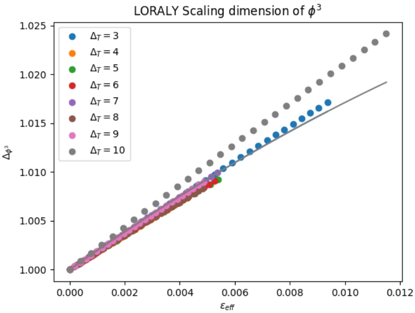
$$\begin{cases} \Delta_{\phi^3} = 1 - \Delta_\phi^{(1)} & \text{(LORI)} \\ \Delta_{\phi^2} = 1 - \Delta_\phi^{(1)} & \text{(LORALY)} \\ \Delta_{\partial\phi} = \Delta_\phi^{(1)} + 1 & \text{existence of some descendants} \end{cases} \tag{3.10}$$

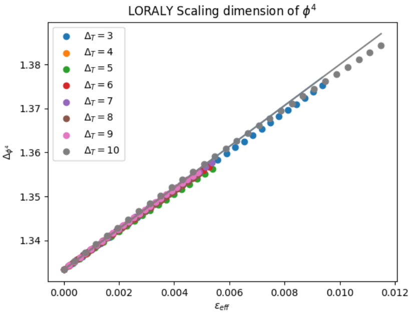
Numerical results for LORI



Solid lines: ε-expansion prediction; points: numerical Hamiltonian truncation.

Numerical results for LORALY





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Comparison with Rayleigh-Schrödinger perturbation theory

- Weak coupling limit $\rightarrow$ comparison with known perturbative expansions in QM

$$\Delta_{\vec{k}} = \Delta_{\vec{k}}^{(0)} + \langle \vec{k}|V|\vec{k}\rangle + \sum_{\vec{l}\neq\vec{k}} \frac{\langle \vec{k}|V|\vec{l}\rangle\langle \vec{l}|V|\vec{k}\rangle}{\Delta_{\vec{k}} - \Delta_{\vec{l}}} + O(V^3) \quad (3.11)$$

- All first order corrections for ϕ^{2n} theory

$$\Delta_{\vec{k}} = \Delta_{\vec{k}}^{(0)} + 2g \sum_{\sum m_r = n} \prod_r \frac{1}{m_r!^2} \left[\frac{(2\Delta_\phi)_r}{r!}\right]^{m_r} (k_r - m_r + 1)_{m_r} + O(g_{2n}^2) \quad (3.12)$$

Comparison with Rayleigh-Schrödinger perturbation theory

- Can conduct the previous tuning procedure to conformal fixed point

$$\begin{cases} \Delta_{\mathbb{1}} = O(g_{2n}^2) \\ \Delta_{\phi} = \frac{1}{2n} - \frac{\varepsilon}{2n} + O(g_{2n}^2) \\ \Delta_{\phi^{2n-1}} = \frac{2n-1}{2n}(1-\varepsilon) + \frac{2g_{2n}}{n!^2}\frac{(2n-1)!}{(n-1)!} + O(g_{2n}^2) \\ \Delta_{\partial^m \phi} = \frac{1-\varepsilon}{2n} + m + O(g_{2n}^2) \end{cases} \tag{3.13}$$

- Solution to (3.13) $\rightarrow$ fixed point couplings. Counter-terms are $O(\varepsilon^2)$ and

$$g_{2n}^* = \frac{n!^2(n-1)!}{2(2n-1)!}\varepsilon + O(\varepsilon^2) \tag{3.14}$$

- Recover $\lambda_* \sim \varepsilon/3$ for LORI $\rightarrow$ same as ε-expansion

Comparison with Rayleigh-Schrödinger perturbation theory

- Can solve for all first order corrections in one-dimensional ϕ^{2n} theory, including LORI:

$$\Delta_{(\partial^l \phi)^k} = k\left(\frac{1}{2n} + l\right) + \left[\frac{k!\Gamma\left(l+\frac{1}{n}\right)^n (n-1)!}{l!^n (k-n)!\Gamma\left(\frac{1}{n}\right)^n (2n-1)!} - \frac{k}{2n}\right]\varepsilon + O(\varepsilon^2)\,, \tag{3.15}$$

- More challenging for LORALY. Need $O(g^2)$ corrections, which are very sensitive to truncation and involve sums over a plethora of states

Effective Hamiltonians

Integrating out high energy modes

- Assuming $\phi = \phi_l + \phi_h$ for some UV cutoff Λ

$$
\begin{aligned}
Z &= \int \mathcal{D}[\phi] e^{-S[\phi]} = \int \mathcal{D}[\phi_l, \phi_h] e^{-S[\phi_l, \phi_h]} \\
&= \int \mathcal{D}[\phi_l] e^{-S_{\text{eff}}[\phi_l]}
\end{aligned} \tag{4.1}
$$

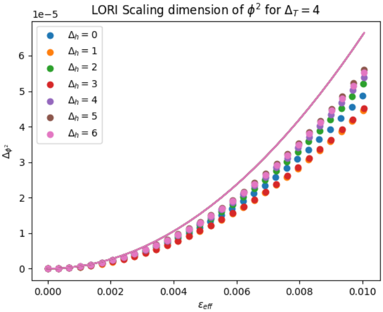
- We know that in general $S_{\text{eff}}[\phi_l] \neq S[\phi_l]$
- Our previous approach is loosely akin to saying $S_{\text{eff}}[\phi_l] = S[\phi_l]$
- Analogously, one should construct an effective Hamiltonian H_{eff}

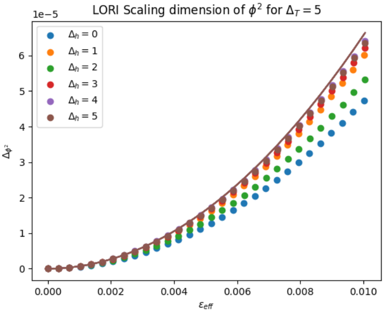
Effective Hamiltonian

$$\langle f|H_{\text{eff}}|i\rangle = \langle f|H_0|i\rangle + \langle f|V|i\rangle + \sum_{\Delta_j > \Delta_T} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{\Delta_f - \Delta_j} + O(V^3) \tag{4.2}$$

- Improves accuracy at low cutoff $\rightarrow$ good for diagonalization
- Can be extended systematically, higher orders listed in Cohen et al. 2022; Miró and Ingoldby 2023
- In principle as many constructions as there are schemes for quantum mechanical perturbation theory (e.g. Schrieffer-Wolff)

Example: improved agreement with perturbation theory in LORI





Solid lines: ε-expansion prediction; scatter plots: effective Hamiltonian truncation.

Include heavy states for next-to-leading order computation for $\Delta_T \leq \Delta \leq \Delta_T + \Delta_h$.

UV divergences in the effective theory

- Using radial quantization, can show that the following diverges

$$\langle 0|H_{\text{eff }2}|0\rangle := -g^2 \sum_{\Delta_{\vec{k}}>\Delta_T} \frac{|\langle \vec{k}| : \phi_L^n(0) + \phi_R^n(0) : |0\rangle|^2}{\Delta_{\vec{k}}} \quad (4.3)$$

- Using explicit matrix elements

$$\langle 0|H_{\text{eff }2}|0\rangle \supset 2n! g^2 \int_0^1 dx x^{n\Delta_\phi - 1}(1-x)^{-2n\Delta_\phi} \propto \Gamma\left(1 - 2n\Delta_\phi\right) \quad (4.4)$$

- Diverges when $n\Delta_\phi \geq \frac{1}{2}$
- Renormalized effective Hamiltonian developed in Miró and Ingoldby 2023 and used for minimal models in Delouche, Elias Miro, and Ingoldby 2024

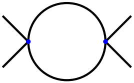
Conclusion

- 1D LORI and LORALY are useful benchmarks for Hamiltonian truncation
- Effective Hamiltonian improves convergence even at low cutoff
- Could push ε-expansion further for more systematic comparisons. E.g. 3-loop results exist for LORI (Benedetti et al. 2020)
- Try to improve numerics for LORALY $\rightarrow$ sign problem?
- Try to extract LORI and LORALY OPE coefficients using Hamiltonian truncation
- **Goal**: connect Hamiltonian truncation to bootstrap results.

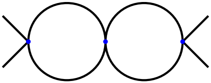
Thank you!
Questions?

Renormalization of LORI

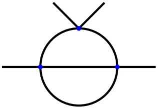
- One and two-loop corrections to ϕ^4 vertex:



One-loop correction



Two-loop correction



Two-loop correction

Two-loop β-function

$$\beta_\lambda(\lambda) = -\varepsilon\lambda + 3\lambda^2 - 35.4851\lambda^3 + O(\lambda^4) \tag{5.1}$$

Fixed point

$$\lambda_* = \frac{\varepsilon}{3} + 1.31426\varepsilon^2 + O(\varepsilon^3) \tag{5.2}$$

Some physical observables in LORI

- Some exact, some at at one, some at two loops

$$\Delta_\phi = \frac{1}{4} - \frac{\varepsilon}{4}\,, \tag{5.3}$$

$$\Delta_{\phi^3} = 1 - \Delta_\phi = \frac{3}{4} + \frac{\varepsilon}{4}\,, \tag{5.4}$$

$$\Delta_{\phi^4} = 1 + \left.\frac{\partial \beta_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_*} = 1 + \varepsilon - 2.94279\varepsilon^2 + O(\varepsilon^3)\,, \tag{5.5}$$

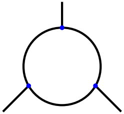
$$\Delta_{\phi^2} = \frac{1}{2} - \frac{\varepsilon}{6} + 0.657131\varepsilon^2 + O(\varepsilon^3)\,, \tag{5.6}$$

$$\forall m \in \mathbb{N}, \Delta_{\partial^m \phi} = m + \Delta_\phi = \frac{1+4m}{4} - \frac{\varepsilon}{4}\,, \tag{5.7}$$

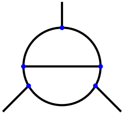
$$\forall n \in \mathbb{N}, \Delta_{\phi^n} = \frac{n}{4} + \frac{n(2n-5)}{12}\varepsilon + O(\varepsilon^2)\,. \tag{5.8}$$

Renormalization of LORALY

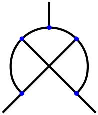
- One and two-loop corrections to ϕ^3 vertex:



One-loop correction



Two-loop correction



Two-loop correction

Two-loop β-function

$$\beta_g(g) = -\varepsilon g + 31.7995 g^3 - 14521 g^5 + O(g^5) \qquad (5.9)$$

Fixed point

$$g_*^2 = 0.031447\varepsilon + 0.451582\varepsilon^3 + O(\varepsilon^3)$$

Some physical observables in LORALY

- Some exact, some at at one, some at two loops

$$\Delta_\phi = \frac{1}{3} - \frac{\varepsilon}{3}\,, \tag{5.10}$$

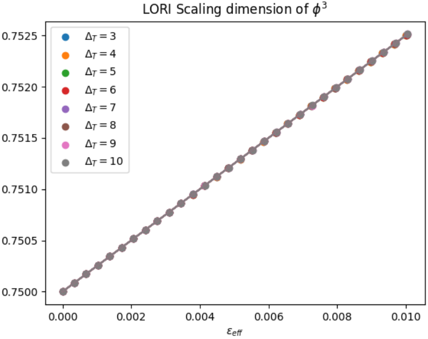
$$\Delta_{\phi^2} = 1 - \Delta_\phi = \frac{2}{3} + \frac{\varepsilon}{3}\,, \tag{5.11}$$

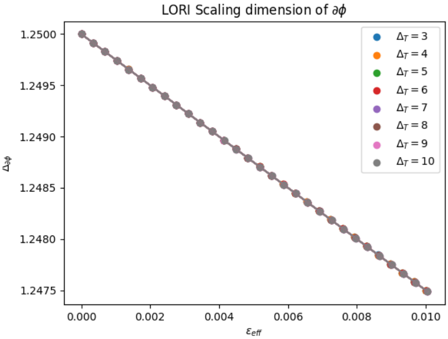
$$\Delta_{\phi^3} = 1 + \left.\frac{\partial \beta_g}{\partial g}\right|_{g=g_*} = 1 + 2\varepsilon - 28.7202\varepsilon^2 + O(\varepsilon^3)\,, \tag{5.12}$$

$$\forall m \in \mathbb{N}, \Delta_{\partial^m \phi} = m + \Delta_\phi = \frac{1+4m}{4} - \frac{\varepsilon}{4}\,, \tag{5.13}$$

$$\forall n \in \mathbb{N}, \Delta_{\phi^n} = \frac{n}{3} + \frac{n(3n-5)}{6}\varepsilon + O(\varepsilon^2)\,. \tag{5.14}$$

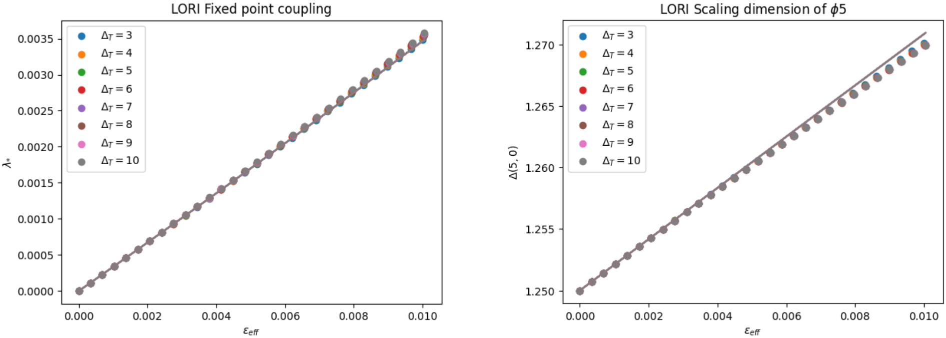
Conformality constraints for LORI





Solid lines: ε-expansion prediction; points: numerical Hamiltonian truncation.

Additional results for LORI



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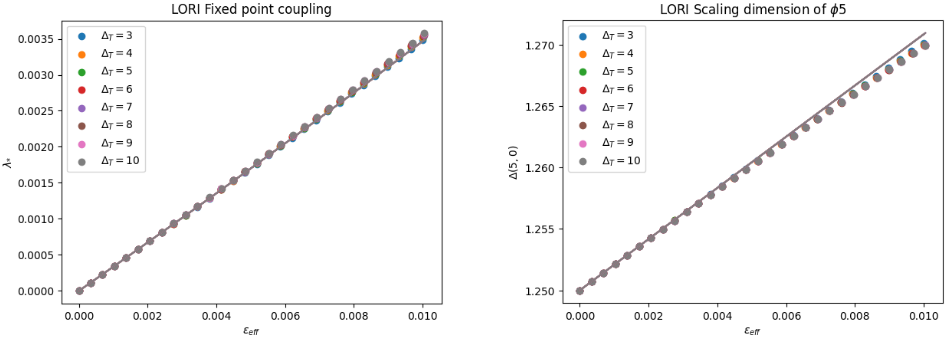

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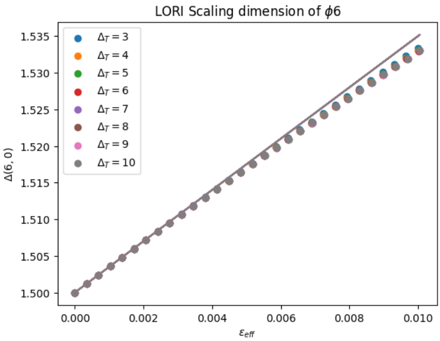

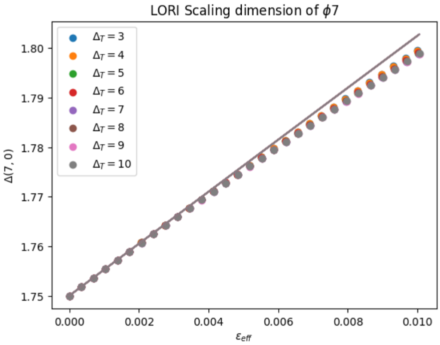

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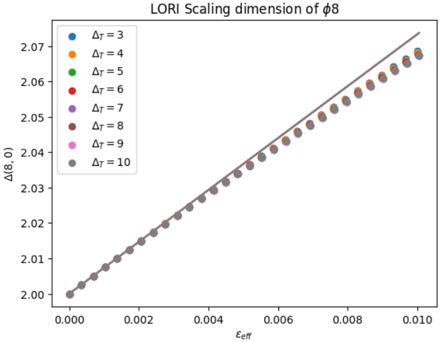
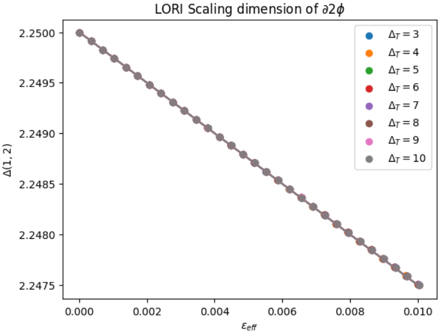
Additional results for LORI





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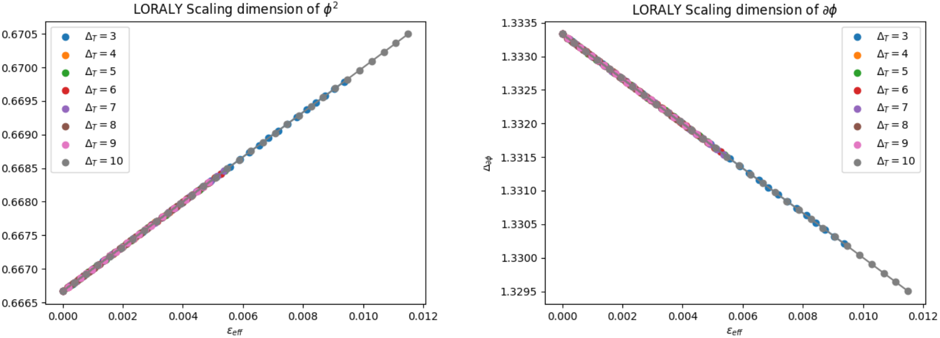
Conformality constraints for LORALY



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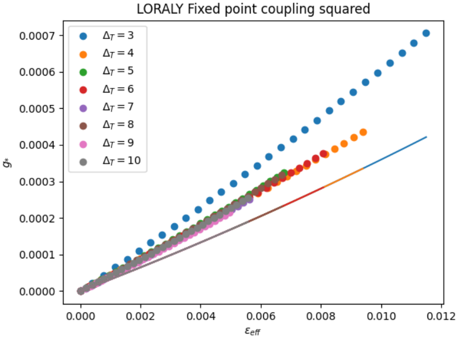
Additional results for LORALY



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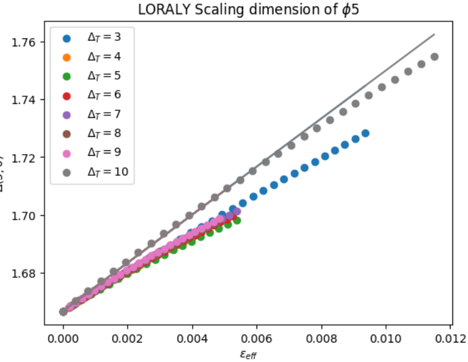

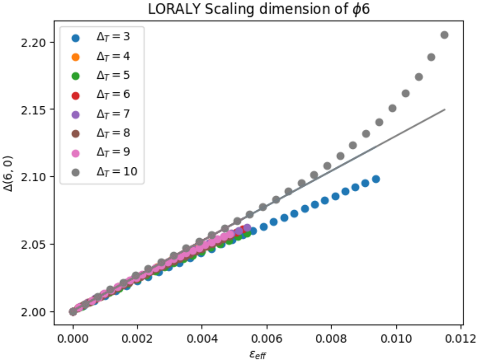

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Matching observables

How do we go about building an effective Hamiltonian? Following Cohen et al. 2022, match physical quantity in effective and fundamental theories.

- Define an effective Hamiltonian with $H_k = O(V^k)$:

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots\,, \tag{5.15}$$

- S-matrix:

$$\langle f|\Sigma(\epsilon)|i\rangle := \lim_{t_f \to \infty} \langle f|U_I(t_f, 0)|i\rangle\,. \tag{5.16}$$

- Transition matrix $T \to$ observable to be matched between effective and fundamental theory:

$$\langle f|\Sigma|i\rangle = \delta_{fi} + \frac{\langle f|T|i\rangle}{\Delta_{fi} + i\epsilon}\,, \tag{5.17}$$

Matching observables

- Adiabatically turn off interactions

$$H = H_0 + Ve^{-\epsilon t} \tag{5.18}$$

- Time-dependent perturbation theory Dyson series:

$$U_I(t_f, 0) = T \exp\left(-i \int_0^{t_f} dt H_I(t)\right) \tag{5.19}$$

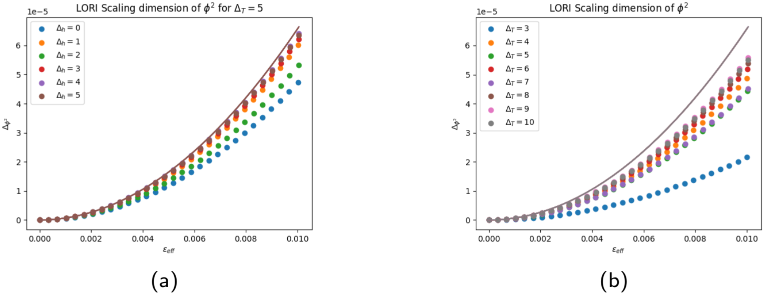
- Expanding and taking matrix elements in the fundamental theory:

$$\langle f|T|i\rangle = \langle f|V|i\rangle + \sum_\alpha \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{\Delta_{f\alpha} + i\epsilon} + O(V^3)\,. \tag{5.20}$$

- Same in the effective theory:

$$\langle f|T|i\rangle = \langle f|H_1|i\rangle + \langle f|H_2|i\rangle + \sum_{\Delta_\alpha \le \Delta_T} \frac{\langle f|H_1|\alpha\rangle\langle\alpha|H_1|i\rangle}{\Delta_{f\alpha} + i\epsilon} + O(V^3)\,. \tag{5.21}$$

Example: improved agreement with perturbation theory in LORI



Solid lines: ε-expansion prediction; scatter plots: effective Hamiltonian truncation (a) naive truncation (b). The effective Hamiltonian at lower truncation can yield better results!

UV divergences: general picture

- E.g. vacuum energy. Assume interaction Lagrangian $\sim g\mathcal{O}(x)$.

$$E_0 = -\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int \prod_{i=1}^{n-1} d^d x_i |x_i|^{\Delta - d} \langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_{n-1}) \mathcal{O}(1) \rangle_0^c \,. \tag{5.22}$$

- n-th order correction diverges for

$$\Delta \geq \frac{n-1}{n} d \tag{5.23}$$

and all corrections diverge at marginality.

UV divergences in the effective theory

- Using radial quantization, can show that the following diverges

$$\langle 0|H_{\text{eff }2}|0\rangle := -g^2 \sum_{\Delta_{\vec{k}}>\Delta_T} \frac{|\langle \vec{k}| : \phi_L^n(0) + \phi_R^n(0) : |0\rangle|^2}{\Delta_{\vec{k}}} . \tag{5.24}$$

- Using explicit matrix elements

$$\langle \vec{k}| : \phi_L^n(0)+\phi_R^n(0) : |0\rangle = 2n! \prod_r \sqrt{\frac{1}{k_r!}\left(\frac{(2\Delta_\phi)_r}{r!}\right)^{k_r}} , \tag{5.25}$$

- which leads to

$$\langle 0|H_{\text{eff }2}|0\rangle = 4n! g^2 \sum_{N=0}^{\infty} \frac{a_{2N}}{n\Delta_\phi + 2N} , \tag{5.26}$$

$$\forall N \in \mathbb{N}, \quad a_N := n! \sum_{\substack{\sum k_r r = 2N \\ \sum k_r = n}} \prod_r \frac{1}{k_r!}\left(\frac{(2\Delta_\phi)_r}{r!}\right)^{k_r} , \tag{5.27}$$

UV divergences in the effective theory

$$\begin{aligned}\langle 0|H_{\text{eff }2}|0\rangle &= 4n!g^2\sum_{N=0}^{\infty}\frac{(2n\Delta_\phi)_{2N}}{(2N)!(n\Delta_\phi+2N)}\\ &= 4n!g^2\int_0^1 dx\sum_{N=0}^{\infty}\frac{(2n\Delta_\phi)_{2N}}{(2N)!}x^{n\Delta_\phi+2N-1}\\ &= 2n!g^2\int_0^1 dx x^{n\Delta_\phi-1}\left[(1-x)^{-2n\Delta_\phi}+(1+x)^{-2n\Delta_\phi}\right].\end{aligned} \tag{5.28}$$

- Recover same type of divergence as that obtained via vacuum bubble graphs!

$$\langle 0|H_{\text{eff }2}|0\rangle \supset 2n!g^2\int_0^1 dx x^{n\Delta_\phi-1}(1-x)^{-2n\Delta_\phi}\propto\Gamma\left(1-2n\Delta_\phi\right). \tag{5.29}$$

A general prescription for renormalization

Steps from Miró and Ingoldby 2023:

- Start from the bare Hamiltonian, which contains, $H_0 + V$.
- Introduce a UV regulator ϵ:

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle \to \left\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \prod_{i<j} \theta(|x_i - x_j| - \epsilon) \right\rangle \tag{5.30}$$

- Absorb ϵ divergences in counter-terms to order M in ϵ:

$$H = H_0 + V \to H = H_0 + V(\epsilon) \tag{5.31}$$

- Write the effective theory to order M. Arrive at some

$$H_0 + K(\epsilon) \underset{\epsilon \to 0}{\to} H_0 + K \text{ finite} \tag{5.32}$$

Renormalized effective theory for 1D CFT

- Start from the (Hermitian conjugate of) next-to-leading order effective Hamiltonian

$$\langle f|H_{\text{eff}}^{(2)}|i\rangle = g^2 \sum_h \frac{\langle f|\phi_\Delta(0)|h\rangle\langle h|\phi_\Delta(0)|i\rangle}{\Delta_{ih}}$$
$$= -g^2 \int_{-\infty}^{0} d\tau \sum_h e^{\tau\Delta_{hi}} \langle f|\phi_\Delta(0)|h\rangle\langle h|\phi_\Delta(0)|i\rangle \,. \tag{5.33}$$

- Time evolve using exponential factors and go to the plane:

$$\langle f|H_{\text{eff}}^{(2)}|i\rangle = -2g^2 \int_{-1}^{1} dx|x|^{\Delta-1} \sum_h \langle f|\phi_\Delta(1)|h\rangle\langle h|\phi_\Delta(x)|i\rangle \,. \tag{5.34}$$

Renormalized effective theory for 1D CFT

- Ignore $\sum |h\rangle\langle h|$ for now and insert complete set of operators (radial quantization relative to 1) and single out $|1-x|<1$ region:

$$\langle f|H_{\text{eff}}^{(2)}|i\rangle \supset -2g^2 \sum_{\mathcal{O}} \int_0^1 dx |x|^{\Delta-1} \langle \mathcal{O}_f(\infty)\mathcal{O}(1)\mathcal{O}_i(0)\rangle\langle \mathcal{O}(\infty)\phi_\Delta(1)\phi_\Delta(x)\rangle \tag{5.35}$$

- x-dependent 3-point function $\propto |1-x|^{-2\Delta-\Delta_{\mathcal{O}}}$ might diverge as $x\to 1 \to$ introduce ϵ regulator:

$$\langle f|H_{\text{eff}}^{(2)}|i\rangle \supset -2g^2 \sum_{\mathcal{O}} \langle f|\mathcal{O}(1)|i\rangle \int_0^{1-\epsilon} dx |x|^{\Delta-1} \langle \mathcal{O}(\infty)\phi_\Delta(1)\phi_\Delta(x)\rangle \tag{5.36}$$

Renormalized effective theory for 1D CFT

- To reintroduce sum over heavy states, need to account for scaling dimension of $|i\rangle \to$ sum over $|h\rangle$ such that $\Delta_h > \Delta_T - \Delta_i$.
- Expanding three-point function, reintroducing pseudo-complete set over heavy states and performing x integration:

$$H^{(2)}_{\text{eff}}(\epsilon)_{fi} = -2g^2 \sum_{\mathcal{O}} \langle f|\mathcal{O}(1)|i\rangle \lambda_{\mathcal{O}\Delta\Delta} \sum_{2n>\Delta'_{T,i}} \frac{[1+(1-\epsilon)^{2n+\Delta}](2\Delta-\Delta_{\mathcal{O}})_{2n}}{(2n)!(2n+\Delta)} + O(\epsilon^0)\,. \tag{5.37}$$

where $\Delta'_{T,i} := \Delta_T - \Delta_i - \Delta$ depends on the initial state.

Renormalized effective theory for 1D CFT

- **Step 1**: introduce local counter-terms to full Hamiltonian, defining $g^{\mathcal{O}} = g^{\mathcal{O}}_{\text{ren}} + g^{\mathcal{O}}_{\text{ct}}(\epsilon)$:

$$H_0 + V \to H_0 + V + \sum_{2\Delta - \Delta_{\mathcal{O}} \geq 1} g^{\mathcal{O}} : (\mathcal{O}_L(0) + \mathcal{O}_R(0)) : \quad (5.38)$$

- Choose counter-term so as to absorb ϵ poles from effective theory

$$\begin{aligned} g_{\text{ct}}(\epsilon) &= g^2 \int_0^{1-\epsilon} dx |x|^{\Delta-1} \langle \mathcal{O}(\infty) \phi_\Delta(1) \phi_\Delta(x) \rangle \\ &= g^2 \lambda_{\mathcal{O}\Delta\Delta} \sum_{n=0}^{\infty} \frac{[1 + (1-\epsilon)^{2n+\Delta}](2\Delta - \Delta_{\mathcal{O}})_{2n}}{(2n)!(2n+\Delta)} . \end{aligned} \quad (5.39)$$

Renormalized effective theory for 1D CFT

- **Step 2**: Truncate to some cutoff Δ_T and introduce next-to-leading order effective correction corresponding to the theory with counter-terms:

$$
\begin{aligned}
(H_{\text{eff }2})_{fi} = &\sum_{2\Delta-\Delta_{\mathcal{O}}>1} g_{\text{ren}}^{\mathcal{O}} \langle f|\mathcal{O}(1)|i\rangle \\
&+ 2g^2 \sum_{2\Delta-\Delta_{\mathcal{O}}\geq 1} \langle f|\mathcal{O}(1)|i\rangle \lambda_{\mathcal{O}\Delta\Delta} \sum_{0\leq 2n\leq \Delta'_{T,i}} \frac{(2\Delta-\Delta_{\mathcal{O}})_{2n}}{(2n)!(2n+\Delta)} + \dots
\end{aligned}
\tag{5.40}
$$

- **Step 3**: Can now safely take $\epsilon \to 0$ limit. Arrive at $H_0 + K$ with

$$
K_2 = \sum_{2\Delta-\Delta_{\mathcal{O}}\geq 1} \langle f|\mathcal{O}(1)|i\rangle \left(g_{\text{ren}}^{\mathcal{O}} + 2g^2 \lambda_{\mathcal{O}\Delta\Delta} \sum_{0\leq 2n\leq \Delta'_{T,i}} \frac{(2\Delta-\Delta_{\mathcal{O}})_{2n}}{(2n)!(2n+\Delta)} \right) + \dots \tag{5.41}
$$